

SUBPROJECTIVE BANACH SPACES

TIMUR OIKHBERG AND EUGENIU SPINU

ABSTRACT. A Banach space X is called subprojective if any of its infinite dimensional subspaces Y contains a further infinite dimensional subspace complemented in X . This paper is devoted to systematic study of subprojectivity. We examine the stability of subprojectivity of Banach spaces under various operations, such as direct or twisted sums, tensor products, and forming spaces of operators. Along the way, we obtain new classes of subprojective spaces.

1. INTRODUCTION AND MAIN RESULTS

We examine various aspects of subprojectivity. Throughout this note, all Banach spaces are assumed to be infinite dimensional, and subspaces, infinite dimensional and closed, until specified otherwise.

A Banach space X is called *subprojective* if every subspace $Y \subset X$ contains a further subspace $Z \subset Y$, complemented in X . This notion was introduced in [40], in order to study the (pre)adjoints of strictly singular operators. Recall that an operator $T \in B(X, Y)$ is *strictly singular* ($T \in \mathcal{SS}(X, Y)$) if T is not an isomorphism on any subspace of X . In particular, it was shown that, if Y is subprojective, and, for $T \in B(X, Y)$, $T^* \in \mathcal{SS}(Y^*, X^*)$, then $T \in \mathcal{SS}(X, Y)$.

Later, connections between subprojectivity and perturbation classes were discovered. More specifically, denote by $\Phi_+(X, Y)$ the set of *upper semi-Fredholm operators* – that is, operators with closed range, and finite dimensional kernel. If $\Phi_+(X, Y) \neq \emptyset$, we define the *perturbation class*

$$P\Phi_+(X, Y) = \{S \in B(X, Y) : T + S \in \Phi_+(X, Y) \text{ whenever } T \in \Phi_+(X, Y)\}.$$

It is known that $\mathcal{SS}(X, Y) \subset P\Phi_+(X, Y)$. In general, this inclusion is proper. However, we get $\mathcal{SS}(X, Y) = P\Phi_+(X, Y)$ if Y is subprojective (see [1, Theorem 7.51] for this, and for similar connections to inessential operators).

Several classes of subprojective spaces are described in [16]. Common examples of non-subprojective space are $L_1(0, 1)$ (since all Hilbertian subspaces of L_1 are not complemented), $C(\Delta)$, where Δ is the Cantor set, or ℓ_∞ (for the same reason). The disc algebra is not subprojective, since by e.g. [41, III.E.3] it contains a copy of $C(\Delta)$. By [40], $L_p(0, 1)$ is subprojective if and only if $2 \leq p < \infty$. Consequently, the Hardy space H_p on the disc is subprojective for exactly the same values of p . Indeed, H_∞ contains the disc algebra. For $1 < p < \infty$, H_p is isomorphic to L_p . The space H_1

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contains isomorphic copies of L_p for $1 < p \leq 2$ [42, Section 3]. On the other hand, VMO is subprojective ([29], see also [36] for non-commutative generalizations).

We start our paper by collecting various facts needed to study subprojectivity (Section 2). Along the way, we prove that subprojectivity is stable under suitable direct sums (Proposition 2.1). However, subprojectivity is not a 3-space property (Proposition 2.7). Consequently, subprojectivity is not stable under the gap metric (Proposition 2.8). Considering the place of subprojective spaces in Gowers dichotomy, we observe that each subprojective space has a subspace with an unconditional basis. However, we exhibit a space with an unconditional basis, but with no subprojective subspaces (Proposition 2.10).

In Section 3, we investigate the subprojectivity of tensor products, and of spaces of operators. A general result on tensor products (Theorem 3.1) yields the subprojectivity on $\ell_p \check{\otimes} \ell_q$ and $\ell_p \hat{\otimes} \ell_q$ for $1 \leq p, q < \infty$ (Corollary 3.3), as well as of $\mathcal{K}(L_p, L_q)$ for $1 < p \leq 2 \leq q < \infty$ (Corollary 3.4). We also prove that the space $B(X)$ is never subprojective (Theorem 3.9), and give an example of non-subprojective tensor product $\ell_2 \otimes_\alpha \ell_2$ (Proposition 3.8).

Throughout Section 4, we work with $C(K)$ spaces, with K compact metrizable. We begin by observing that $C(K)$ is subprojective if and only if K is scattered. Then we prove that $C(K, X)$ is subprojective if and only if both $C(K)$ and X are (Theorem 4.1). Turning to spaces of operators, we show that, for K scattered, $\Pi_{qp}(C(K), \ell_q)$ is subprojective (Proposition 4.4). Then we study continuous fields on a scattered base space, proving that any scattered separable CCR C^* -algebra is subprojective (Corollary 4.7).

Section 5 shows that, in many cases, subprojectivity passes from a sequence space to the associated Schatten spaces (Proposition 5.1).

Proceeding to Banach lattices, in Section 6 we prove that p -disjointly homogeneous p -convex lattices ($2 \leq p < \infty$) are subprojective (Proposition 6.2). In Section 7 (Proposition 7.1), we show that the lattice $\widetilde{X(\ell_p)}$ is subprojective whenever X is. Consequently (Proposition 7.3), if X is a subprojective space with an unconditional basis and non-trivial cotype, then $\text{Rad}(X)$ is subprojective.

Throughout the paper, we use the standard Banach space results and notation. By $B(X, Y)$ and $\mathcal{K}(X, Y)$ we denote the sets of linear bounded and compact operators, respectively, acting between Banach spaces X and Y . $\mathbf{B}(X)$ refers to the closed unit ball of X . For $p \in [1, \infty]$, we denote by p' the “adjoint” of p (that is, $1/p + 1/p' = 1$).

2. GENERAL FACTS ABOUT SUBPROJECTIVITY

We begin this section by showing that subprojectivity passes to direct sums.

Proposition 2.1. *(a) Suppose X and Y are Banach spaces. Then the following are equivalent:*

- (1) *Both X and Y are subprojective.*
- (2) *$X \oplus Y$ is subprojective.*

(b) Suppose X_1, X_2, \dots are Banach spaces, and \mathcal{E} is a space with a 1-unconditional basis. Then the following are equivalent:

- (1) *The spaces $\mathcal{E}, X_1, X_2, \dots$ are subprojective.*

(2) $(\sum_n X_n)_\mathcal{E}$ is subprojective.

In (b), we view \mathcal{E} as a space of sequences of scalars, equipped with the norm $\|\cdot\|_\mathcal{E}$. $(\sum_n X_n)_\mathcal{E}$ refers to the space of all sequences $(x_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} X_n$, endowed with the norm $\|(x_n)_{n \in \mathbb{N}}\| = \|(\|x_n\|_{X_n})\|_\mathcal{E}$. Due to the 1-unconditionality (actually, 1-suppression unconditionality suffices), $(\sum_n X_n)_\mathcal{E}$ is a Banach space.

We begin by making two simple observations, to be used several times throughout this paper.

Proposition 2.2. *Consider Banach spaces X and X' , and $T \in B(X, X')$. Suppose Y is a subspace of X , $T|_Y$ is an isomorphism, and $T(Y)$ is complemented in X' . Then Y is complemented in X .*

Proof. If Q is a projection from X' to $T(Y)$, then $T^{-1}QT$ is a projection from X onto Y . ■

This immediately yields:

Corollary 2.3. *Suppose X and X' are Banach spaces, and X' is subprojective. Suppose, furthermore, that Y is a subspace of X , and there exists $T \in B(X, X')$ so that $T|_Y$ is an isomorphism. Then Y contains a subspace complemented in X .*

The following version of “Principle of Small Perturbations” is folklore, and essentially contained in [5]. We include the proof for the sake of completeness.

Proposition 2.4. *Suppose (x_k) is a seminormalized basic sequence in a Banach space X , and (y_k) is a sequence so that $\lim_k \|x_k - y_k\| = 0$. Suppose, furthermore, that every subspace of $\text{span}[y_k : k \in \mathbb{N}]$ contains a subspace complemented in X . Then $\text{span}[x_k : k \in \mathbb{N}]$ contains a subspace complemented in X .*

Proof. Replacing x_k by $x_k/\|x_k\|$, we can assume that (x_k) normalized. Denote the biorthogonal functionals by x_k^* , and set $K = \sup_k \|x_k^*\|$. Passing to a subsequence, we can assume that $\sum_k \|x_k - y_k\| < 1/(2K)$. Define the operator $U \in B(X)$ by setting $Ux = \sum_k x_k^*(x)(y_k - x_k)$. Clearly $\|U\| < 1/2$, and therefore, $V = I_X + U$ is invertible. Furthermore, $Vx_k = y_k$. If Q is a projection from X onto a subspace $W \subset \text{span}[y_k : k \in \mathbb{N}]$, then $P = V^{-1}QV$ is a projection from X onto a subspace $Z \subset \text{span}[x_k : k \in \mathbb{N}]$. ■

Remark 2.5. Note that, in the proof above, the kernels and the ranges of the projections Q and P are isomorphic, via the action of V .

Proof of Proposition 2.1. It is easy to see that subprojectivity is inherited by subspaces. Thus, in both (a) and (b), only the implication (2) \Rightarrow (1) needs to be established.

(a) Throughout the proof, P_X and P_Y stand for the coordinate projections from $X \oplus Y$ onto X and Y , respectively. We have to show that any subspace E of $X \oplus Y$ contains a further subspace G , complemented in $X \oplus Y$.

Show first that E contains a subspace F so that either $P_X|_F$ or $P_Y|_F$ is an isomorphism. Indeed, suppose $P_X|_F$ is not an isomorphism, for any such F . Then $P_X|_E$ is strictly singular, hence there exists a subspace $F \subset E$, so that $P_X|_F$ has norm less than $1/2$. But $P_X + P_Y = I_{X \oplus Y}$, hence, by the triangle inequality,

$\|P_Y f\| \geq \|f\| - \|P_X f\| \geq \|f\|/2$ for any $f \in F$. Consequently, $P_Y|_F$ is an isomorphism.

Thus, by passing to a subspace, and relabeling if necessary, we can assume that E contains a subspace F , so that $P_X|_F$ is an isomorphism. By Corollary 2.3, F contains a subspace G , complemented in X .

Set $F' = P_X(F)$, and let V be the inverse of $P_X : F \rightarrow F'$. By the subprojectivity of X , F' contains a subspace G' , complemented in X via a projection Q . Then $P = VQP_X$ gives a projection onto $G = V(G') \subset F$.

(b) Here, we denote by P_n the coordinate projection from $X = (\sum_k X_k)_\mathcal{E}$ onto X_n . Furthermore, we set $Q_n = \sum_{k=1}^n P_k$, and $Q_n^\perp = \mathbf{1} - Q_n$. We have to show that any subspace $Y \subset X$ contains a subspace Y_0 , complemented in X . To this end, consider two cases.

(i) For some n , and some subspace $Z \subset Y$, $Q_n|_Z$ is an isomorphism. By part (a), $X_1 \oplus \dots \oplus X_n = Q_n(X)$ is subprojective. Apply Corollary 2.3 to obtain Y_0 .

(ii) For every n , $Q_n|_Y$ is not an isomorphism – that is, for every $n \in \mathbb{N}$, and every $\varepsilon > 0$, there exists a norm one $y \in Y$ so that $\|Q_n y\| < \varepsilon$. Therefore, for every sequence of positive numbers (ε_i) , we can find $0 = N_0 < N_1 < N_2 < \dots$, and a sequence of norm one vectors $y_i \in Y$, so that, for every i , $\|Q_{N_i} y_i\|, \|Q_{N_{i+1}}^\perp y_i\| < \varepsilon_i$. By a small perturbation principle, we can assume that Y contains norm one vectors (y'_i) so that $Q_{N_i} y'_i = Q_{N_{i+1}}^\perp y'_i = 0$ for every i . Write $y'_i = (z_j)_{j=N_{i+1}}^{N_{i+1}+1}$, with $z_j \in X_j$. Then $Z = \text{span}[(0, \dots, 0, z_j, 0, \dots) : j \in \mathbb{N}]$ (z_j is in j -th position) is complemented in X . Indeed, if $z_j \neq 0$, find $z_j^* \in X_j^*$ so that $\|z_j^*\| = \|z_j\|^{-1}$, and $\langle z_j^*, z_j \rangle = 1$. If $z_j = 0$, set $z_j^* = 0$. For $x = (x_j)_{j \in \mathbb{N}} \in X$, define $Rx = (\langle z_j^*, x_j \rangle z_j)_{j \in \mathbb{N}}$. It is easy to see that R is a projection onto Z , and $\|R\|$ does not exceed the unconditionality constant of \mathcal{E} .

Now note that $J : Z \rightarrow \mathcal{E} : (\alpha_1 z_1, \alpha_2 z_2, \dots) \mapsto (\alpha_1 \|z_1\|, \alpha_2 \|z_2\|, \dots)$ is an isometry. Let $Y' = \text{span}[y'_i : i \in \mathbb{N}]$, and $Y_\mathcal{E} = J(Y')$. By the subprojectivity of \mathcal{E} , $Y_\mathcal{E}$ contains a subspace W , which is complemented in \mathcal{E} via a projection R_1 . Then $J^{-1}R_1JR$ is a projection from X onto $Y_0 = J^{-1}(W) \subset X$. \blacksquare

Remark 2.6. From the last proposition it follows the (strong) p -sum of subprojective Banach spaces is subprojective. On the other hand, the infinite weak sum of subprojective spaces need not be subprojective.

Recall that if X is a Banach space, then

$$\ell_p^{weak}(X) = \{x = (x_n)_{n=1}^\infty \in X \times X \times X \dots : \sup_{x^* \in X^*} (\sum |x^*(x_n)|^p)^{\frac{1}{p}} < \infty\}.$$

It is known that $\ell_p^{weak}(X)$ is isomorphic to $B(\ell_{p'}, X)$ ($\frac{1}{p} + \frac{1}{p'} = 1$), see [9, Theorem 2.2]. We show that, for $X = \ell_r$ ($r \geq p'$), $B(\ell_{p'}, X)$ contains a copy of ℓ_∞ , and therefore, is not subprojective. To this end, denote by (e_i) and (f_i) the canonical bases in ℓ_r and $\ell_{p'}$ respectively. For $\alpha = (\alpha_i) \in \ell_\infty$, define $B(\ell_{p'}, X) \ni U\alpha : e_i \mapsto \alpha_i f_i$. Clearly, U is an isomorphism.

Note that the situation is different for $r < p'$. Then, by Pitt's Theorem, $B(\ell_{p'}, \ell_r) = \mathcal{K}(\ell_{p'}, \ell_r)$. In the next section we prove that the latter space is subprojective.

Next we show that subprojectivity is not a 3-space property.

Proposition 2.7. *For $1 < p < \infty$ there exists a non-subprojective Banach space Z_p , containing a subspace X_p , so that X_p and Z_p/X_p are isomorphic to ℓ_p .*

Proof. [23, Section 6] gives us a short exact sequence

$$0 \longrightarrow \ell_p \xrightarrow{j_p} Z_p \xrightarrow{q_p} \ell_p \longrightarrow 0,$$

where the injection j_p is strictly cosingular, and the quotient map q_p is strictly singular. By [23, Theorem 6.2], Z_p is not isomorphic to ℓ_p . By [23, Theorem 6.5], any non-strictly singular operator on Z_p fixes a copy of Z_p . Consequently, $j_p(\ell_p)$ contains no complemented subspaces (by [26, Theorem 2.a.3], any complemented subspace of ℓ_p is isomorphic to ℓ_p). ■

It is easy to see that subprojectivity is stable under isomorphisms. However, it is not stable under a rougher measure of “closeness” of Banach spaces – the gap measure. If Y and Z are subspaces of a Banach space X , we define the *gap* (or *opening*)

$$\Theta_X(Y, Z) = \max \left\{ \sup_{y \in Y, \|y\|=1} \text{dist}(y, Z), \sup_{z \in Z, \|z\|=1} \text{dist}(z, Y) \right\}.$$

We refer the reader to the comprehensive survey [32] for more information. Here, we note that Θ_X satisfies a “weak triangle inequality”, hence it can be viewed as a measure of closeness of subspaces. The following shows that subprojectivity is not stable under Θ_X .

Proposition 2.8. *There exists a Banach space X with a subprojective subspace Y so that, for every $\varepsilon > 0$, X contains a non-subprojective space Z with $\Theta_X(Y, Z) \leq \varepsilon$.*

Proof. Our Y will be isomorphic to ℓ_p , where $p \in (1, \infty)$ is fixed. By Proposition 2.7, there exists a non-subprojective Banach space W , containing a subspace W_0 , so that both W_0 and $W' = W/W_0$ are isomorphic to ℓ_p . Denote the quotient map $W \rightarrow W'$ by q . Consider $X = W \oplus_1 W'$ and $Y = W_0 \oplus_1 W' \subset X$. Furthermore, for $\varepsilon > 0$, define $Z_\varepsilon = \{\varepsilon w \oplus_1 qw : w \in W\}$. Clearly, Y is isomorphic to $\ell_p \oplus \ell_p \sim \ell_p$, hence subprojective, while Z_ε is isomorphic to W , hence not subprojective. By [32, Lemma 5.9], $\Theta_X(Y, Z_\varepsilon) \leq \varepsilon$. ■

Looking at subprojectivity through the lens of Gowers dichotomy and observing that a subprojective Banach space does not contain hereditarily indecomposable subspaces, we immediately obtain the following.

Proposition 2.9. *Every subprojective space has a subspace with an unconditional basis.*

The converse to the above proposition is false.

Proposition 2.10. *There exists a Banach space with an unconditional basis, without subprojective subspaces.*

Proof. In [18, Section 5], T. Gowers and B. Maurey construct a Banach space X with a 1-unconditional basis, so that any operator on X is a strictly singular perturbation of a diagonal operator. We prove that X has no subprojective subspaces. In doing so, we are re-using the notation of that paper. In particular, for $n \in \mathbb{N}$ and $x \in X$,

we define $\|x\|_{(n)}$ as the supremum of $\sum_{i=1}^n \|x_i\|$, where x_1, \dots, x_n are successive vectors so that $x = \sum_i x_i$. It is known that, for every block subspace Y in X , every $c > 1$, and every $n \in \mathbb{N}$, there exists $y \in Y$ so that $1 = \|y\| \leq \|y\|_{(n)} < c$. This technical result can be used to establish a remarkable property of X : suppose Y is a subspace of X , with a normalized block basis (y_k) . Then any zero-diagonal (relative to the basis (y_k)) operator on Y is strictly singular. Consequently, any $T \in B(Y)$ can be written as $T = \Lambda + S$, where Λ is diagonal, and S is zero-diagonal, hence strictly singular. This result is proved in [18] for $Y = X$, but an inspection yields the generalization described above.

Suppose, for the sake of contradiction, that X contains a subprojective subspace Y . A small perturbation argument shows we can assume Y to be a block subspace. Blocking further, we can assume that Y is spanned by a block basis (y_j) , so that $1 = \|y_j\| \leq \|y_j\|_{(j)} < 1 + 2^{-j}$. We achieve the desired contradiction by showing that no subspace of $Z = \text{span}[y_1 + y_2, y_3 + y_4, \dots]$ is complemented in Y .

Suppose P is an infinite rank projection from Y onto a subspace of Z . Write $P = \Lambda + S$, where S is a strictly singular operator with zeroes on the main diagonal, and $\Lambda = (\lambda_j)_{j=1}^\infty$ is a diagonal operator (that is, $\Lambda y_j = \lambda_j y_j$ for any j). As $\sup_j \|y_j\|_{(j)} < \infty$, by [18, Section 5] we have $\lim_j S y_j = 0$. Note that $(\Lambda + S)^2 = \Lambda + S$, hence $\text{diag}(\lambda_j^2 - \lambda_j) = \Lambda^2 - \Lambda = S - \Lambda S - S\Lambda - S^2$ is strictly singular, or equivalently, $\lim_j \lambda_j(1 - \lambda_j) = 0$. Therefore, there exists a 0–1 sequence (λ'_j) so that $\Lambda' - \Lambda$ is compact (equivalently, $\lim_j(\lambda_j - \lambda'_j) = 0$), where $\Lambda' = \text{diag}(\lambda'_j)$ is a diagonal projection. Then $P = \Lambda' + S'$, where $S' = S + (\Lambda - \Lambda')$ is strictly singular, and satisfies $\lim_j S' y_j = 0$. The projection P is not strictly singular (since it is of infinite rank), hence $\Lambda' = P - S'$ is not strictly singular. Consequently, the set $J = \{j \in \mathbb{N} : \lambda'_j = 1\}$ is infinite.

Now note that, for any j , $\|P y_j - y_j\| \geq 1/2$. Indeed, $P y_j \in Z$, hence we can write $P y_j = \sum_k \alpha_k (y_{2k-1} + y_{2k})$. Let $\ell = \lceil j/2 \rceil$. By the 1-unconditionality of our basis, $\|y_j - P y_j\| \geq \|y_j - \alpha_\ell (y_{2\ell-1} + y_{2\ell})\| \geq \max\{|1 - \alpha_\ell|, |\alpha_\ell|\} \geq 1/2$. For $j \in J$, $S' y_j = P y_j - y_j$, hence $\|S' y_j\| \geq 1/2$, which contradicts $\lim_j \|S' y_j\| = 0$. ■

Remark 2.11. The preceding statement provides an example of an atomic order continuous Banach lattice without subprojective subspaces. One can also observe that if a Banach lattice is not order continuous, then it contains a subprojective subspace c_0 . Also, if a Banach lattice is non-atomic order continuous with an unconditional basis, then it contains a subprojective subspace ℓ_2 (i.e. [24, Theorem 2.3]).

Finally, one might ask whether, in the definition of subprojectivity, the projections from X onto Z can be uniformly bounded. More precisely, we call a Banach space X *uniformly subprojective* (with constant C) if, for every subspace $Y \subset X$, there exists a subspace $Z \subset Y$ and a projection $P : X \rightarrow Z$ with $\|P\| \leq C$. The proof of [16, Proposition 2.4] essentially shows that the following spaces are uniformly subprojective: (i) ℓ_p ($1 \leq p < \infty$) and c_0 ; (ii) the Lorentz sequence spaces $\mathfrak{l}_{p,w}$; (iii) the Schreier space; (iv) the Tsirelson space; (v) the James space. Additionally, $L_p(0, 1)$ is uniformly subprojective for $2 \leq p < \infty$. This can be proved by combining Kadets-Pelczynski dichotomy with the results of [2] about the existence of “nicely complemented” copies of ℓ_2 . Moreover, any c_0 -saturated separable space is uniformly subprojective, since any isomorphic copy of c_0 contains a λ -isomorphic copy of c_0 , for

any $\lambda > 1$ [26, Proposition 2.e.3]. By Sobczyk's Theorem, a λ -isomorphic copy of c_0 is 2λ -complemented in every separable superspace. In particular, if K is a countable metric space, then $C(K)$ is uniformly subprojective [12, Theorem 12.30].

However, in general, subprojectivity need not be uniform. Indeed, suppose $2 < p_1 < p_2 < \dots < \infty$, and $\lim_n p_n = \infty$. By Proposition 2.1(b), $X = (\sum_n L_{p_n}(0, 1))_2$ is subprojective. The span of independent Gaussian random variables in L_p (which we denote by G_p) is isometric to ℓ_2 . Therefore, by [17, Corollary 5.7], any projection from L_p onto G_p has norm at least $c_0\sqrt{p}$, where c_0 is a universal constant. Thus, X is not uniformly subprojective.

3. SUBPROJECTIVITY OF TENSOR PRODUCTS AND SPACES OF OPERATORS

Suppose X_1, X_2, \dots, X_k are Banach spaces with unconditional FDD, implemented by finite rank projections $(P'_{1n}), (P'_{2n}), \dots, (P'_{kn})$, respectively. That is, $P'_{in}P'_{im} = 0$ unless $n = m$, $\lim_N \sum_{n=1}^N P'_{in} = I_{X_i}$ point-norm, and $\sup_{N, \pm} \|\sum_{n=1}^N \pm P'_{in}\| < \infty$ (this quantity is sometimes referred to as *the FDD constant of X_i*). Let $E_{in} = \text{ran}(P'_{in})$.

We say that a sequence $(w_j)_{j=1}^\infty \subset X_1 \otimes X_2 \otimes \dots \otimes X_k$ is *block-diagonal* if there exists a sequence $0 = N_1 < N_2 < \dots$ so that

$$w_j \in \left(\sum_{n=N_j+1}^{N_{j+1}} E_{1n} \right) \otimes \left(\sum_{n=N_j+1}^{N_{j+1}} E_{2n} \right) \otimes \dots \otimes \left(\sum_{n=N_j+1}^{N_{j+1}} E_{kn} \right).$$

Suppose \mathcal{E} is an unconditional sequence space, and $\tilde{\otimes}$ is a tensor product of Banach spaces. The Banach space $X_1 \tilde{\otimes} X_2 \tilde{\otimes} \dots \tilde{\otimes} X_k$ is said to *satisfy the \mathcal{E} -estimate* if there exists a constant $C \geq 1$ so that, for any block diagonal sequence $(w_j)_{j \in \mathbb{N}}$ in $X_1 \tilde{\otimes} X_2 \tilde{\otimes} \dots \tilde{\otimes} X_k$, we have

$$(3.1) \quad C^{-1} \|(\|w_j\|)_{j \in \mathbb{N}}\|_{\mathcal{E}} \leq \left\| \sum_j w_j \right\| \leq C \|(\|w_j\|)_{j \in \mathbb{N}}\|_{\mathcal{E}}$$

Theorem 3.1. *Suppose X_1, X_2, \dots, X_k are subprojective Banach spaces with unconditional FDD, and $\tilde{\otimes}$ is a tensor product. Suppose, furthermore, that for any finite increasing sequence $\mathbf{i} = [1 \leq i_1 < \dots < i_\ell \leq k]$, there exists an unconditional sequence space $\mathcal{E}_{\mathbf{i}}$, so that $X_{i_1} \tilde{\otimes} X_{i_2} \tilde{\otimes} \dots \tilde{\otimes} X_{i_\ell}$ satisfies the $\mathcal{E}_{\mathbf{i}}$ -estimate. Then $X_1 \tilde{\otimes} X_2 \tilde{\otimes} \dots \tilde{\otimes} X_k$ is subprojective.*

A similar result for ideals of operators holds as well. We keep the notation for projections implementing the FDD in Banach spaces X_1 and X_2 . We say that a Banach operator ideal \mathcal{A} is *suitable* (for the pair (X_1, X_2)) if the finite rank operators are dense in $\mathcal{A}(X_1, X_2)$ (in its ideal norm). We say that a sequence $(w_j)_{j \in \mathbb{N}} \subset \mathcal{A}(X_1, X_2)$ is *block diagonal* if there exists a sequence $0 = N_1 < N_2 < \dots$ so that, for any j , $w_j = (P_{2, N_j} - P_{2, N_{j-1}})w_j(P_{1, N_j} - P_{1, N_{j-1}})$. If \mathcal{E} is an unconditional sequence space, we say that $\mathcal{K}(X_1, X_2)$ *satisfies the \mathcal{E} -estimate* if, for some constant C ,

$$(3.2) \quad C^{-1} \|(\|w_j\|)_{j \in \mathbb{N}}\|_{\mathcal{E}} \leq \left\| \sum_j w_j \right\|_{\mathcal{A}} \leq C \|(\|w_j\|)_{j \in \mathbb{N}}\|_{\mathcal{E}}$$

holds for any finite block-diagonal sequence (w_j) .

Theorem 3.2. *Suppose X_1 and X_2 are Banach spaces with unconditional FDD, so that X_1^* and X_2 are subprojective. Suppose, furthermore, that the ideal \mathcal{A} is suitable for (X_1, X_2) , and $\mathcal{A}(X_1, X_2)$ satisfies the \mathcal{E} -estimate for some unconditional sequence \mathcal{E} . Then $\mathcal{A}(X_1, X_2)$ is subprojective.*

Before proving these theorems, we state a few consequences.

Corollary 3.3. *The spaces $X_1 \tilde{\otimes} \dots \tilde{\otimes} X_n$ and $X_1 \hat{\otimes} \dots \hat{\otimes} X_n$ are subprojective where X_i is either isomorphic to ℓ_{p_i} ($1 \leq p_i < \infty$) or c_0 for every $1 \leq i \leq n$.*

For $n = 2$, this result goes back to [37] and [31] (the injective and projective cases, respectively).

Suppose a Banach space X has an FDD implemented by projections (P'_n) – that is, $P'_n P'_m = 0$ unless $n = m$, $\sup_{N, \pm} \|\sum_{n=1}^N \pm P'_n\| < \infty$, and $\lim_N \sum_{n=1}^N P'_n = I_X$ point-norm. We say that X satisfies the *lower p -estimate* if there exists a constant C so that, for any finite sequence $\xi_j \in \text{ran } P_j$, $\|\sum_j \xi_j\|^p \geq C \sum_j \|\xi_j\|^p$. The smallest C for which the above inequality holds is called the *lower p -estimate constant*. The *upper p -estimate*, and the *upper p -estimate constant*, are defined in a similar manner. Note that, if X is an unconditional sequence space, then the above definitions coincide with the standard one (see e.g. [27, Definition 1.f.4]).

Corollary 3.4. *Suppose the Banach spaces X_1 and X_2 have unconditional FDD, satisfy the lower and upper p -estimates respectively, and both X_1^* and X_2 are subprojective. Then $\mathcal{K}(X_1, X_2)$ is subprojective.*

Before proceeding, we mention several instances where the above corollary is applicable. Note that, if X has type 2 (cotype 2), then X satisfies the upper (resp. lower) 2-estimate. Indeed, suppose X has type 2, and w_1, \dots, w_n are such that $w_j = P_j w_j$ for any j . Then

$$\left\| \sum_j w_j \right\| \leq C \text{Ave}_\pm \left\| \sum_j \pm w_j \right\| \leq C T_2(X) \left(\sum_j \|w_j\|^2 \right)^{1/2}$$

($T_2(X)$ is the type 2 constant of X). The cotype case is handled similarly. Thus, we can state:

Corollary 3.5. *Suppose the Banach spaces X_1 and X_2 have unconditional FDD, cotype 2 and type 2 respectively, and both X_1^* and X_2 are subprojective. Then $\mathcal{K}(X_1, X_2)$ is subprojective.*

This happens, for instance, if $X_1 = L_p(\mu)$ or \mathfrak{C}_p ($1 < p \leq 2$) and $X_2 = L_q(\mu)$ or \mathfrak{C}_q ($2 \leq q < \infty$). Indeed, the type and cotype of these spaces are well known (see e.g. [35]). The Haar system provides an unconditional basis for L_p . The existence of unconditional FDD of \mathfrak{C}_p spaces is given by [4].

Proof of Theorem 3.1. We will prove the theorem by induction on k . Clearly, we can take $k = 1$ as the basic case. Suppose the statement of the theorem holds for a tensor product of any $k - 1$ subprojective Banach spaces that satisfy \mathcal{E} -estimate. We will show that the statement holds for the tensor product of k Banach spaces $X = X_1 \tilde{\otimes} X_2 \tilde{\otimes} \dots \tilde{\otimes} X_k$.

For notational convenience, let $P_{in} = \sum_{k=1}^n P'_{ik}$, and $I_i = I_{X_i}$. If $A \in B(X)$ is a projection, we use the notation A^\perp for $I_X - A$. Furthermore, define the projections $Q_n = P_{1n} \otimes P_{2n} \otimes \dots \otimes P_{kn}$ and $R_n = P_{1n}^\perp \otimes P_{2n}^\perp \otimes \dots \otimes P_{kn}^\perp$. Renorming all X_i 's if necessary, we can assume that their unconditional FDD constants equal 1.

First show that, for any n , $\text{ran } R_n^\perp$ is subprojective. To this end, write $R_n^\perp = \sum_{i=1}^k P^{(i)}$, where the projections $P^{(i)}$ are defined by

$$\begin{aligned} P^{(1)} &= P_{1n} \otimes I_2 \otimes \dots \otimes I_k, \\ P^{(2)} &= P_{1n}^\perp \otimes P_{2n} \otimes I_3 \otimes \dots \otimes I_k, \\ P^{(3)} &= P_{1n}^\perp \otimes P_{2n}^\perp \otimes P_{3n} \otimes I_4 \otimes \dots \otimes I_k, \\ &\vdots \quad \vdots \quad \vdots \\ P^{(k)} &= P_{1n}^\perp \otimes P_{2n}^\perp \otimes \dots \otimes P_{k-1,n}^\perp \otimes P_{kn} \end{aligned}$$

(note also that $P^{(i)}P^{(j)} = 0$ unless $i = j$). Thus, there exists i so that $P^{(i)}$ is an isomorphism on a subspace $Y' \subset Y$. Now observe that the range of $P^{(i)}$ is isomorphic to a subspace of $\ell_\infty^N(X^{(i)})$, where $N = \text{rank } P_{in}$, and

$$X^{(i)} = X_1 \tilde{\otimes} X_2 \tilde{\otimes} \dots \tilde{\otimes} X_{i-1} \tilde{\otimes} X_{i+1} \tilde{\otimes} \dots \tilde{\otimes} X_k.$$

By the induction hypothesis, $X^{(i)}$ is subprojective. By Proposition 2.1, $\text{ran } P^{(i)}$ is subprojective for every i , hence so is $\text{ran } R_n^\perp$.

Now suppose Y is an infinite dimensional subspace of X . We have to show that Y contains a subspace Z , complemented in X . If there exists $n \in \mathbb{N}$ so that $R_n^\perp|_Y$ is not strictly singular, then, by Corollary 2.3, Z contains a subspace complemented in X .

Now suppose $R_n^\perp|_Z$ is strictly singular for any n . It is easy to see that, for any sequence of positive numbers (ε_m) , one can find $0 = n_0 < n_1 < n_2 < \dots$, and norm one elements $x_m \in Y$, so that, for any m , $\|R_{n_{m-1}}^\perp x_m\| + \|x_m - Q_{n_m} x_m\| < \varepsilon_m$. By a small perturbation, we can assume that $x_m = R_{n_{m-1}}^\perp Q_{n_m} x_m$. That is,

$$x_m \in \text{ran} \left((P_{1,n_m} - P_{1,n_{m-1}}) \otimes (P_{2,n_m} - P_{2,n_{m-1}}) \otimes \dots \otimes (P_{k,n_m} - P_{k,n_{m-1}}) \right).$$

Let $E_{im} = \text{ran} (P_{i,n_m} - P_{i,n_{m-1}})$, and $W = \text{span}[E_{1m} \otimes E_{2m} \otimes \dots \otimes E_{km} : m \in \mathbb{N}] \subset X$. Applying ‘‘Tong’s trick’’ (see e.g. [26, p. 20]), and taking the 1-unconditionality of our FDDs into account, we see that

$$U : X \rightarrow W : x \mapsto \sum_m ((P_{1,n_m} - P_{1,n_{m-1}}) \otimes \dots \otimes (P_{k,n_m} - P_{k,n_{m-1}}))x$$

defines a contractive projection onto W . Furthermore, $Z = \text{span}[x_m : m \in \mathbb{N}]$ is complemented in W . Indeed, the projection $P_{i,n_m} - P_{i,n_{m-1}}$ ($i, m \in \mathbb{N}$) is contractive, hence we can identify $E_{1m} \tilde{\otimes} \dots \tilde{\otimes} E_{km}$ with $(E_{1m} \otimes \dots \otimes E_{km}) \cap X$. By the Hahn-Banach Theorem, for each m there exists a contractive projection U_m on $E_{1m} \tilde{\otimes} \dots \tilde{\otimes} E_{km}$, with range $\text{span}[x_m]$. By our assumption, there exists an unconditional sequence space \mathcal{E} so that $X_1 \tilde{\otimes} \dots \tilde{\otimes} X_k$ satisfies the \mathcal{E} -estimate. Then, for any finite sequence $w_m \in E_{1m} \tilde{\otimes} \dots \tilde{\otimes} E_{km}$, (3.1) yields

$$\left\| \sum_k U_k w_k \right\| \leq C(\|U_k w_k\|)_\mathcal{E} \leq C(\|w_k\|)_\mathcal{E} \leq C^2 \left\| \sum_k U_k w_k \right\|.$$

Thus, Z is complemented in X . ■

Sketch of the proof of Theorem 3.2. On $\mathcal{A}(X_1, X_2)$ we define the projection $R_n : \mathcal{A}(X_1, X_2) \rightarrow \mathcal{A}(X_1, X_2) : w \mapsto P_{2n}^\perp w P_{1n}$. Then the range of R_n^\perp is isomorphic to $X_1^* \oplus \dots \oplus X_1^* \oplus X_2 \oplus \dots \oplus X_2$. Then proceed as in the the proof of Theorem 3.1 (with $k = 2$). \blacksquare

To prove Corollary 3.3, we need two auxiliary results.

Lemma 3.6. *Suppose $1 < p_i < \infty$ ($1 \leq i \leq n$) and $X = \hat{\otimes}_{i=1}^n \ell_{p_i}$.*

- (1) *If $\sum 1/p_i > n-1$, then X satisfies the ℓ_s -estimate with $1/s = \sum 1/p_i - (n-1)$.*
- (2) *If $\sum 1/p_i \leq n-1$, then X satisfies the c_0 -estimate.*

Proof. Suppose (w_j) is a finite block-diagonal sequence in X . We shall show that $\|\sum_j w_j\| = \|(\|w_j\|)\|_s$, with s as in the statement of the lemma. To this end, let (U_{ij}) be coordinate projections on ℓ_{p_i} for every $1 \leq i \leq n$, such that $w_j = U_{1j} \otimes \dots \otimes U_{nj} w_j$, and for each i , $U_{ik} U_{im} = 0$ unless $k = m$. Letting $p'_i = p_i/(p_i - 1)$, we see that

$$\|\sum_j w_j\| = \sup_{\xi_i \in \ell_{p'_i}, \|\xi_i\| \leq 1} \left| \langle \sum_j w_j, \otimes_i \xi_i \rangle \right|.$$

Choose $\otimes_i \xi_i$ with $\|\xi_i\| \leq 1$, and let $\xi_{ij} = U_{ij} \xi_i$. Then $\sum_j \|\xi_{ij}\|^{p'_i} \leq 1$, and

$$\left| \langle \sum_j w_j, \otimes_i \xi_i \rangle \right| \leq \sum_j |\langle w_j, \otimes_i \xi_i \rangle| = \sum_j |\langle w_j, \otimes_i \xi_{ij} \rangle| \leq \sum_j \|w_j\| \Pi_{i=1}^n \|\xi_{ij}\|.$$

Now let $1/r = \sum 1/p'_i = n - \sum 1/p_i$. By Hölder's Inequality,

$$\left(\sum_j \left(\prod_{i=1}^n \|\xi_{ij}\| \right)^r \right)^{1/r} \leq \prod_{i=1}^n \left(\sum_j \|\xi_{ij}\|^{p'_i} \right)^{1/p'_i} \leq 1.$$

If $\sum 1/p_i \leq n-1$, then $r \leq 1$, hence $\sum_j \Pi_{i=1}^n \|\xi_{ij}\| \leq 1$. Therefore, $\|\sum_j w_j\| \leq \max_j \|w_j\| = (\|w_j\|)_{c_0}$. Otherwise, $r > 1$, and

$$\|\sum_j w_j\| \leq \left(\sum_j \|w_j\|^s \right)^{1/s} \left(\sum_j (\Pi_{i=1}^n \|\xi_{ij}\|)^r \right)^{1/r} \leq \left(\sum_j \|w_j\|^s \right)^{1/s} = (\|w_j\|)_s,$$

where $1/s = 1 - 1/r = \sum 1/p_i - n + 1$.

In a similar fashion, we show that $\|\sum_j w_j\| \geq (\|w_j\|)_s$. For $s = \infty$, the inequality $\|\sum_j w_j\| \geq \max_j \|w_j\|$ is trivial. If s is finite, assume $\sum_j \|w_j\|^s = 1$ (we are allowed to do so by scaling). Find norm one vectors $\xi_{ij} \in \ell_{p'_i}$ so that $\xi_{ij} = U_{ij} \xi_i$, and $\|w_j\| = \langle w_j, \otimes_i \xi_{ij} \rangle$. Let $\gamma_j = \|w_j\|^{s/r}$. Then $\sum_j \gamma_j^r = 1 = \sum_j \gamma_j \|w_j\|$. Further, set $\alpha_{ij} = \gamma_j^{\Pi_{l \neq i} p'_l / (\sum_{m=1}^n \Pi_{l \neq m} p'_l)}$. An elementary calculation shows that $\gamma_j = \Pi_{i=1}^n \alpha_{ij}$, and $\sum_j \alpha_{ij}^{p'_i} = 1$. Let $\xi_i = \sum_j \alpha_{ij} \xi_{ij}$. Then $\|\xi_i\|_{p'_i} = 1$, and therefore,

$$\|\sum_j w_j\| \geq \langle \sum_j w_j, \otimes_i \xi_i \rangle = \sum_j \Pi_{i=1}^n \alpha_{ij} \langle w_j, \otimes_i \xi_{ij} \rangle = \sum_j \gamma_j \|w_j\| = 1.$$

This establishes the desired lower estimate. \blacksquare

Lemma 3.7. *For $1 \leq p_i \leq \infty$, $X = \ell_{p_1} \hat{\otimes} \ell_{p_2} \hat{\otimes} \dots \hat{\otimes} \ell_{p_n}$ satisfies the ℓ_r -estimate, where $1/r = \sum 1/p_i$ if $\sum 1/p_i < 1$, and $r = 1$ otherwise. Here, we interpret ℓ_∞ as c_0 .*

Proof. The spaces involved all have the Contractive Projection Property (the identity can be approximated by contractive finite rank projections). Thus, the duality between injective and projective tensor products of finite dimensional spaces (see e.g. [7, Section 1.2.1]) shows that, for $w \in X$,

$$\|w\| = \sup \{ |\langle x, w \rangle| : x \in \ell_{p'_1} \check{\otimes} \dots \check{\otimes} \ell_{p'_n}, \|x\| \leq 1 \}$$

(here, as before, $1/p'_i + 1/p_i = 1$). Abusing the notation somewhat, we denote by P_{im} the projection on the span of the first m basis vectors of both ℓ_{p_i} and $\ell_{p'_i}$. Suppose a finite sequence $(w_k)_{k=1}^N \in X$ is block-diagonal, or more precisely, $w_k = ((P_{1,m_k} - P_{1,m_{k-1}}) \otimes \dots \otimes (P_{n,m_k} - P_{n,m_{k-1}}))w_k$ for every k . Define the operator U on X by setting $Ux = \sum_{k=1}^N ((P_{1,m_k} - P_{1,m_{k-1}}) \otimes \dots \otimes (P_{n,m_k} - P_{n,m_{k-1}}))x$. We also use U_0 to denote the similarly defined operator on X^* . By ‘‘Tong’s trick’’ (see e.g. [26, p. 20]), since X and X^* has an unconditional basis, U (U_0) is a contractive projection onto its range W (W_0). Then

$$\begin{aligned} \left\| \sum_k w_k \right\| &= \sup \left\{ \left| \left\langle \sum_k w_k, x \right\rangle \right| : \|x\|_{X^*} \leq 1 \right\} \\ &= \sup \left\{ \left| \left\langle U \left(\sum_k w_k \right), x \right\rangle \right| : \|x\|_{X^*} \leq 1 \right\} \\ &= \sup \left\{ \left| \left\langle \sum_k w_k, U_0 x \right\rangle \right| : \|x\|_{X^*} \leq 1 \right\}. \end{aligned}$$

Write $U_0 x = \sum_{k=1}^N x_k$. By Lemma 3.6 there is an s (either $1/s = \sum 1/p'_i - (n-1) = 1 - \sum 1/p_i$ or $s = \infty$) $\|(\|x_k\|)\|_s = \|U_0 x\| \leq \|x\| \leq 1$. Moreover,

$$\left\langle \sum_k w_k, U_0 x \right\rangle = \left\langle \sum_k w_k, \sum_k x_k \right\rangle = \sum_k \langle w_k, x_k \rangle,$$

and therefore,

$$\left\| \sum_k w_k \right\| = \sup \left\{ \sum_k |\langle w_k, x_k \rangle| : \|(\|x_k\|)\|_s \leq 1 \right\} = \|(\|w_k\|)\|_r.$$

■

Proof of Corollary 3.3. Combine Theorem 3.1 with Lemma 3.6 and 3.7. ■

Proof of Corollary 3.4. To apply Theorem 3.2, we have to show that $\mathcal{K}(X_1, X_2)$ satisfies the c_0 -estimate. By renorming, we can assume that the FDD constants of X_1 and X_2 equal 1. Suppose $(w_k)_{k=1}^N$ is a block-diagonal sequence, with $w_k = (P_{2,n_k} - P_{2,n_{k-1}})w_k(P_{1,n_k} - P_{1,n_{k-1}})$. Let $w = \sum_k w_k$. Then $\|w\| \geq \|(P_{2,n_k} - P_{2,n_{k-1}})w(P_{1,n_k} - P_{1,n_{k-1}})\| = \|w_k\|$, hence $\|w\| \geq \max_k \|w_k\|$. To prove the reverse inequality (with some constant), pick a norm one $\xi \in X_1$, and let $\xi_k = (P_{1,n_k} - P_{1,n_{k-1}})x$. Then $\eta_k = w\xi_k$ satisfies $(P_{2,n_k} - P_{2,n_{k-1}})\eta_k = \eta_k$. Set $\eta = w\xi = \sum_k \eta_k$. Denote by C_1 (C_2) lower (upper) p -estimate constants of X_1 (resp. X_2). Then

$$\begin{aligned} \|w\xi\|^p &= \|\eta\|^p \leq C_1 \sum_k \|\eta_k\|^p \leq C_2 \sum_k \|w_k\|^p \|\xi_k\|^p \\ &\leq \max_k \|w_k\|_p C_2 \sum_k \|\xi_k\|^p \leq \max_k \|w_k\|^p C_2 C_1 \sum_k \|\xi_k\|^p = C_2 C_1 \|\xi\|^p. \end{aligned}$$

Taking the supremum over all $\xi \in \mathbf{B}(X_1)$, $\|w\| \leq (C_1 C_2)^{1/p} \max_k \|w_k\|$. \blacksquare

In general, a tensor product of subprojective spaces (in fact, of Hilbert spaces) need not be subprojective.

Proposition 3.8. *There exists a tensor norm \otimes_α , so that, for every Banach spaces X and Y , $X \otimes_\alpha Y$ is a Banach space, and $\ell_2 \otimes_\alpha \ell_2$ is not subprojective.*

Proof. Note first that there exists a separable symmetric sequence space \mathcal{E} which is not subprojective. Indeed, let U be the space with an unconditional basis which is complementably universal for all spaces with unconditional bases, see [26, Proposition 2.d.10]. As noted in [26, Section 3.b], this space has a symmetric basis (in fact, uncountably many non-equivalent symmetric bases). On the other hand, U is not subprojective, since it contains a (complemented) copy of L_p for $1 < p < 2$. Renorming U to make its basis 1-symmetric, we obtain \mathcal{E} .

Now suppose X and Y are Banach spaces. For $a \in X \otimes Y$, we set $\|a\|_\alpha = \sup\{\|(u \otimes v)(a)\|_{\mathcal{E}(H,K)}\}$, where the supremum is taken over all contractions $u : X \rightarrow H$ and $v : Y \rightarrow K$ (H and K are Hilbert spaces). Clearly \otimes_α is a norm on $X \otimes Y$. It is easy to see that, for any $a \in X \otimes Y$, $T_X \in B(X, X_0)$, and $T_Y \in B(Y, Y_0)$, $\|(T_X \otimes T_Y)(a)\|_\alpha \leq \|T_X\| \|T_Y\| \|a\|_\alpha$. Consequently, $\|x \otimes y\|_\alpha = \|x\| \|y\|$. Thus, $\|\cdot\|_\alpha$ is indeed a tensor norm (in the sense of e.g. [, Section 12]). We denote by $X \otimes_\alpha Y$ the completion of $X \otimes Y$ in this norm.

If X and Y are Hilbert spaces, then for $a \in X \otimes Y$ we have $\|a\|_\alpha = \|a\|_{\mathcal{E}(X^*, Y)}$. Identifying ℓ_2 with its adjoint, we see that \mathcal{E} embeds into $\ell_2 \otimes_\alpha \ell_2$ as the space of diagonal operators. As \mathcal{E} is not subprojective, neither is $\ell_2 \otimes_\alpha \ell_2$. \blacksquare

Here is another wide class of non-subprojective spaces.

Theorem 3.9. *Let X be an infinite dimensional Banach space. Then $B(X)$ is not subprojective.*

Proof. Suppose, for the sake of contradiction, that $B(X)$ is subprojective. Fix a norm one element $x^* \in X^*$. For $x \in X$ define $T_x \in B(X) : y \mapsto \langle x^*, y \rangle x$. Clearly $M = \{T_x : x \in X\}$ is a closed subspace of $B(X)$, isomorphic to X . Therefore, X is subprojective. By Proposition 2.9, we can find a subspace $N \subset M$ with an unconditional basis. We shall deduce that $B(X)$ contains a copy of ℓ_∞ , which is not subprojective.

If N is not reflexive, then N contains either a copy of c_0 or a copy of ℓ_1 , see [26, Proposition 1.c.13]. By [26, Proposition 2.a.2], any subspace of ℓ_p (c_0) contains a further subspace isomorphic to ℓ_p (resp. c_0) and complemented in ℓ_p (resp. c_0), hence we can pass from N to a further subspace W , isomorphic to ℓ_1 or c_0 , and complemented in X by a projection P . Embed $B(W)$ isomorphically into $B(X)$ by sending $T \in B(W)$ to $PTP \in B(X)$, where P is a projection from X onto W . It is easy to see that $B(W)$ contains subspaces isomorphic to ℓ_∞ , thus, $B(X)$ is not subprojective.

There is only one option left: N is reflexive. Pick a subspace $W \subset N$, complemented in X . It has the Bounded Approximation Property [26, Theorem 1.e.13]. As in the previous paragraph, $B(W)$ embeds isomorphically into $B(X)$. Since $B(W) \neq \mathcal{K}(W)$, [11, Theorem 4(1)] shows that $B(W)$ contains an isomorphic copy of ℓ_∞ . This rules out the subprojectivity of $B(X)$. \blacksquare

Question 3.10. Suppose X is a subprojective Banach space. (i) Is $\text{Rad}(X)$ subprojective? (ii) If $2 \leq p < \infty$, must $L_p(X)$ be subprojective?

Question 3.11. Is a “classical” (injective, projective, etc.) tensor product of subprojective spaces necessarily subprojective? Note that the Fremlin tensor product $\otimes_{|\pi|}$ of Banach lattices (the ordered analogue of the projective product) can destroy subprojectivity. Indeed, by [6], $L_2 \otimes_{|\pi|} L_2$ contains a copy of L_1 . L_2 is clearly subprojective, while L_1 is not (see e.g. [40]).

4. SPACES OF CONTINUOUS FUNCTIONS

In this section we deal with spaces of functions on scattered spaces. Recall that a topological space is *scattered* if every compact subset has an isolated point. It is known that a compact set is scattered and metrizable if and only if it is countable (in this case, $C(K)$, and even its dual, are separable). For more information, see e.g. [12, Section 12]. It is well known that, if K is a compact Hausdorff set, then $C(K)$ is separable if and only if K is metrizable.

If K is countable, then $C(K)$ is c_0 -saturated [12, Section 12], and the copies of c_0 are complemented, by Sobczyk’s Theorem. Otherwise, by Milutin’s Theorem (see e.g. [41, III.D.19], $C(K)$ is isomorphic to $C([0, 1])$. Thus, a separable space $C(K)$ is subprojective if and only if K is scattered.

Furthermore (see e.g. [34]), it is known that K is scattered if and only if it supports no non-zero atomic measures. Then $C(K)^*$ is isometric to $\ell_1(K)$. Otherwise, $C(K)^*$ contains a copy of $L_1(0, 1)$. Thus, $C(K)^*$ is subprojective if and only if K is scattered.

4.1. Tensor products of $C(K)$. In this subsection we study the subprojectivity of projective and injective tensor products of $C(K)$. Our main result is:

Theorem 4.1. *Suppose K is a compact metrizable space, and X is a Banach space. Then the following are equivalent:*

- (1) *K is scattered, and X is subprojective.*
- (2) *$C(K, X)$ is subprojective.*

Proof. The implication (2) \Rightarrow (1) is easy. The space $C(K, X)$ contains copies of $C(K)$ and of X , hence the last two spaces are subprojective. By the preceding paragraph, K must be scattered.

To prove (1) \Rightarrow (2), first fix some notation. Suppose λ is a countable ordinal. We consider the interval $[0, \lambda]$ with the order topology – that is, the topology generated by the open intervals (α, β) , as well as $[0, \beta)$ and $(\alpha, \lambda]$. Abusing the notation slightly, we write $C(\lambda, X)$ for $C([0, \lambda], X)$.

Suppose K is scattered. By [38, Chapter 8], K is isomorphic to $[0, \lambda]$, for some countable limit ordinal λ . Fix a subprojective space X . We use induction on λ to show that, for any countable ordinal λ ,

$$(4.1) \quad C(\lambda, X) \text{ is subprojective.}$$

By Proposition 2.1, (4.1) holds for $\lambda \leq \omega$ (indeed, c is isomorphic to c_0 , hence $c(X) = c \otimes X$ is isomorphic to $c_0(X) = c_0 \otimes X$). Let \mathcal{F} denote the set of all countable ordinals for which (4.1) fails. If \mathcal{F} is non-empty, then it contains a minimal element, which we denote by μ . Note that μ is a limit ordinal. Indeed, otherwise it has

an immediate predecessor $\mu - 1$. It is easy to see that $C(\mu, X)$ is isomorphic to $C(\mu - 1, X) \oplus X$, hence, by Proposition 2.1, $C(\mu - 1, X)$ is not subprojective. Let $C_0(\mu, X) = \{f \in C(\mu, X) : \lim_{\nu \rightarrow \mu} f(\nu) = 0\}$. Clearly $C(\mu, X)$ is isomorphic to $C_0(\mu, X) \oplus X$, hence we obtain the desired contradiction by showing that $C_0(\mu, X)$ is subprojective.

To do this, suppose Y is a subspace of $C_0(\mu, X)$, so that no subspace of Y is complemented in $C_0(\mu, X)$. For $\nu < \mu$, define the projection $P_\nu : C(\mu, X) \rightarrow C(\nu, X) : f \mapsto f \mathbf{1}_{[0, \nu]}$. If, for some $\nu < \mu$ and some subspace $Z \subset Y$, $P_\nu|_Z$ is an isomorphism, then Z contains a subspace complemented in X , by the induction hypothesis and Corollary 2.3. Now suppose $P_\nu|_Y$ is strictly singular for any ν . We construct a sequence of “almost disjoint” elements of Y . To do this, take an arbitrary y_1 from the unit sphere of Y . Pick $\nu_1 < \mu$ so that $\|y_1 - P_{\nu_1} y_1\| < 10^{-1}$. Now find a norm one $y_2 \in Y$ so that $\|P_{\nu_1} y_2\| < 10^{-2}/2$. Proceeding further in the same manner, we find a sequence of ordinals $0 = \nu_0 < \nu_1 < \nu_2 < \dots$, and a sequence of norm one elements $y_1, y_2, \dots \in Y$, so that $\|y_k - z_k\| < 10^{-k}$, where $z_k = (P_{\nu_k} - P_{\nu_{k-1}})y_k$. The sequence (z_k) is equivalent to the c_0 basis, and the same is true for the sequence (y_k) .

Moreover, $\text{span}[z_k : k \in \mathbb{N}]$ is complemented in $C(\mu, X)$. Indeed, let $\nu = \sup_k \nu_k$. We claim that $\mu = \nu$. If $\nu < \mu$, then P_ν is an isomorphism on $\text{span}[y_k : k \in \mathbb{N}]$, contradicting our assumption. Let $W_k = (P_{\nu_k} - P_{\nu_{k-1}})(C_0(X))$, and find a norm one linear functional w_k so that $w_k(z_k) = \|z_k\|$. Define

$$Q : C_0(\mu, X) \rightarrow C_0(\mu, X) : f \mapsto \sum_k w_k((P_{\nu_k} - P_{\nu_{k-1}})f) z_k.$$

Note that $\lim_k \|(P_{\nu_k} - P_{\nu_{k-1}})f\| = 0$, hence the range of Q is precisely the span of the elements z_k . By Small Perturbation Principle, Y contains a subspace complemented in $C_0(\mu, X)$. \blacksquare

The above theorem shows that $C(K) \hat{\otimes} X$ is subprojective if and only if both $C(K)$ and X are. We do not know whether a similar result holds for other tensor products. We do, however, have:

Proposition 4.2. *Suppose K is a compact metrizable space, and W is either ℓ_p ($1 \leq p < \infty$) or c_0 . Then $C(K) \hat{\otimes} W$ is subprojective if and only if K is scattered.*

Proof. Clearly, if K is not scattered, then $C(K)$ is not subprojective. So suppose K is scattered. We deal with the case of $W = \ell_p$, as the c_0 case is handled similarly. As before, we can assume that $K = [0, \lambda]$, where λ is a countable ordinal. We use transfinite induction on λ . The base case is easy: if λ is a finite ordinal, then $C(\lambda) \hat{\otimes} \ell_p = \ell_\infty^N \hat{\otimes} \ell_p$ is subprojective. Furthermore the same is true for $\lambda = \omega$ (then $C(\lambda) = c$).

Suppose, for the sake of contradiction, that λ is the smallest countable ordinal so that $C(\lambda) \hat{\otimes} \ell_p$ is not subprojective. Reasoning as before, we conclude that λ is a limit ordinal. Furthermore, $C(\lambda) \sim C_0(\lambda)$, hence $C_0(\lambda) \hat{\otimes} \ell_p$ is not subprojective.

Denote by $Q_n : \ell_p \rightarrow \ell_p$ the projection on the first n basis vectors in ℓ_p , and let $Q_n^\perp = I - Q_n$. For $f \in C_0(\lambda)$ and an ordinal $\nu < \lambda$, define $P_\nu f = \chi_{[0, \nu]} f$, and $P_\nu^\perp = I - P_\nu$.

Suppose X is a subspace of $C_0(\lambda) \hat{\otimes} \ell_p$ which has no subspaces complemented in $C_0(\lambda) \hat{\otimes} \ell_p$. By the induction hypothesis, $(P_\nu \otimes I_{\ell_p})|_X$ is strictly singular for any

$\nu < \lambda$. Furthermore, $(I_{C_0(\lambda)} \otimes Q_n)|_Y$ must be strictly singular. Indeed, otherwise Y has a subspace Z so that $(I_{C_0(\lambda)} \otimes Q_n)|_Z$ is an isomorphism, whose range is subprojective (the range of $I_{C_0(\lambda)} \otimes Q_n$ is isomorphic to the sum of n copies of $C(\lambda)$, hence subprojective). Therefore, for any $\nu < \lambda$ and $n \in \mathbb{N}$, $(I - P_\nu^\perp \otimes Q_n^\perp)|_Y$ is strictly singular. Therefore we can find a normalized basis (x_i) in Y , and sequences $0 = \nu_0 < \nu_1 < \dots < \lambda$, and $0 = n_0 < n_1 < \dots$, so that $\|x_i - (P_{\nu_{i-1}}^\perp \otimes Q_{n_{i-1}}^\perp)x_i\| < 10^{-3i}/2$. By passing to a further subsequence, we can assume that $\|(P_{\nu_i} \otimes Q_{n_i})x_i\| < 10^{-3i}/2$. Thus, by the Small Perturbation Principle, it suffices to show the following statement: If (y_i) is a normalized sequence in $C_0(\lambda) \hat{\otimes} \ell_p$, so that there exist non-negative integers $0 = n_0 < n_1 < n_2 < \dots$, and ordinals $0 = \nu_0 < \nu_1 < \nu_2 < \dots < \lambda$, with the property that $y_i = ((P_{\nu_i} - P_{\nu_{i-1}}) \otimes (Q_{n_i} - Q_{n_{i-1}}))y_i$ for any i , then $Y = \text{span}[y_i : i \in \mathbb{N}]$ is contractively complemented in $C(K) \hat{\otimes} \ell_p$.

Denote by X the span of all x 's for which there exists an i so that $x = ((P_{\nu_i} - P_{\nu_{i-1}}) \otimes (Q_{n_i} - Q_{n_{i-1}}))x$. Then Y is contractively complemented in $C(K) \hat{\otimes} \ell_p$. In fact, we can define a contractive projection onto X as follows. Suppose first $u = \sum_{j=1}^N a_j \otimes b_j$, with b_i 's having finite support in ℓ_p . Then set $Pu = \sum_{i=1}^\infty ((P_{\nu_i} - P_{\nu_{i-1}}) \otimes (Q_{n_i} - Q_{n_{i-1}}))u$. Due to our assumption on the b_i 's, there exists M so that $Pu = \sum_{i=1}^M ((P_{\nu_i} - P_{\nu_{i-1}}) \otimes (Q_{n_i} - Q_{n_{i-1}}))u$. To show that $\|Pu\| \leq \|u\|$, define, for $\varepsilon = (\varepsilon_i)_{i=1}^M \in \{-1, 1\}^M$, the operator of multiplication by $\sum_{i=1}^M M\varepsilon_i \chi_{[\nu_{i-1}+1, \nu_i]}$ on $C_0(\lambda)$. The operator $V_\varepsilon \in B(\ell_p)$ is defined similarly. Both U_ε and V_ε are contractive. Furthermore, $Pu = \text{Ave}_\varepsilon(U_\varepsilon \otimes V_\varepsilon)u$. Therefore, we can use continuity to extend P to a contractive projection from $C_0(\lambda) \hat{\otimes} \ell_p$ onto X .

To construct a contractive projection from X onto Y , we need to show that the blocks of X satisfy the ℓ_p -estimate. That is, if $x_i = ((P_{\nu_i} - P_{\nu_{i-1}}) \otimes (Q_{n_i} - Q_{n_{i-1}}))x_i$ for each i , then $\|\sum_i x_i\|^p = \sum_i \|x_i\|^p$. To this end, use trace duality to identify $(C_0(\lambda) \hat{\otimes} \ell_p)^*$ with $B(\ell_p, \ell_1([0, \lambda)))$. P^* is the “block” projection onto the space of “block diagonal” operators which map the elements of ℓ_p supported on $(n_{i-1}, n_i]$ onto the vectors in ℓ_1 supported on $(\nu_{i-1}, \nu_i]$. If T_i 's are the blocks of such an operator, then $\|\sum_i T_i\|^{p'} = \sum_i \|T_i\|^{p'}$, where $1/p + 1/p' = 1$. (see the proof of Corollary 3.3). By duality, $\|\sum_i x_i\|^p = \sum_i \|x_i\|^p$. \blacksquare

Remark 4.3. Suppose K is a scattered metrizable space. We do not know whether $C(K) \hat{\otimes} C(K)$ is necessarily subprojective. The proof above cannot be emulated directly, since P may not be well-defined. More specifically, we cannot quite define Pu if $u = f \otimes f$, with $f = \chi_{[0, \mu]}$, with $\sup_i \nu_i < \mu < \lambda$.

4.2. Operators on $C(K)$.

Proposition 4.4. *Suppose K is a scattered compact metrizable space, and $1 \leq p \leq q < \infty$. Then the space $\Pi_{qp}(C(K), \ell_q)$ is subprojective.*

Recall that $\Pi_{qp}(X, Y)$ stands for the space of (q, p) -summing operators – that is, the operators for which there exists a constant C so that, for any $x_1, \dots, x_n \in X$,

$$\left(\sum_i \|Tx_i\|^q \right)^{1/q} \leq C \sup_{x^* \in \mathbf{B}(X^*)} \left(\sum_i |x^*(x_i)|^p \right)^{1/p}.$$

The smallest value of C is denoted by $\pi_{pq}(T)$.

Note that, if a compact Hausdorff space K is not scattered, then $C(K)^*$ contains L_1 [34], hence $\Pi_{qp}(C(K), \ell_q)$ is not subprojective.

The following lemma may be interesting in its own right.

Lemma 4.5. *Suppose X is a Banach space, K is a compact metrizable scattered space, and $1 \leq p \leq q < \infty$. Then, for any $T \in \Pi_{qp}(C(K), X)$, and any $\varepsilon > 0$, there exists a finite rank operator $S \in \Pi_{qp}(C(K), X)$ with $\pi_{pq}(T - S) < \varepsilon$.*

In proving Proposition 4.4 and Lemma 4.5, we consider the cases of $p = q$ and $p < q$ separately. If $p = q$, we are dealing with q -summing operators. By Pietsch Factorization Theorem, $T \in B(C(K), X)$ is q -summing if and only if there exists a probability measure μ on K so that T factors as $\tilde{T} \circ j$, where $j : C(K) \rightarrow L_q(\mu)$ is the formal identity, and $\|T\| \leq \pi_q(T)$. Moreover, μ and \tilde{T} can be selected in such a way that $\|\tilde{T}\| = \pi_q(T)$. As K is scattered, there exist distinct points $k_1, k_2, \dots \in K$, and non-negative scalars $\alpha_1, \alpha_2, \dots$, so that $\sum_i \alpha_i = 1$, and $\mu = \sum_i \alpha_i \delta_{k_i}$.

Now suppose $T \in B(C(K), X)$ satisfies $\pi_q(T) = 1$. Keeping the above notation, find $N \in \mathbb{N}$ so that $(\sum_{i=N+1}^{\infty} \alpha_i)^{\frac{1}{q}} < \varepsilon$. Denote by u and v the operators of multiplication by $\chi_{\{k_1, \dots, k_N\}}$ and $\chi_{\{k_{N+1}, k_{N+2}, \dots\}}$, respectively, acting on $L_q(\mu)$. It is easy to see that $\text{rank } u \leq N$, and $\|vj\| < \varepsilon$. Then $S = \tilde{T}uj$ works in Lemma 4.5.

If $1 \leq p < q$, then (see e.g. [9, Chapter 10] or [39, Chapter 21]), $\Pi_{qp}(C(K), X) = \Pi_{q1}(C(K), X)$, with equivalent norms. Henceforth, we set $p = 1$. We have a probability measure μ on K , and a factorization $T = \tilde{T}j$, where $j : C(K) \rightarrow L_{q1}(\mu)$ is the formal identity, and $\tilde{T} : L_{q1}(\mu) \rightarrow X$ satisfies $\|\tilde{T}\| \leq c\pi_{q1}(T)$ (c is a constant depending on q).

In this case, the proof of Lemma 4.5 proceeds as for q -summing operators, except that now, we need to select N so that $c(\sum_{i=N+1}^{\infty} \alpha_i)^{1/q} < \varepsilon$.

Proof of Proposition 4.4. It is well known that, for any T , $\pi_{qp}(T) = \pi_{qp}(T^{**})$. Moreover, by Lemma 4.5, any (q, p) -summing operator on $C(K)$ can be approximated by a finite rank operator. Then we can identify $\Pi_{qp}(C(K), X)$ with the completion of the algebraic tensor product $C(K)^* \otimes X$ in the appropriate tensor norm which we denote by α . Recalling that $C(K)^* = \ell_1$ (the canonical basis in ℓ_1 corresponds to the point evaluation functionals), we can describe α in more detail: for $u = \sum_i a_i \otimes x_i \in \ell_1 \otimes X$, $\|u\|_\alpha = \pi_{qp}(\bar{u})$, where $\bar{u} : \ell_\infty \rightarrow X$ is defined by $\bar{u}b = \sum_i b(a_i)x_i$. Furthermore, by the injectivity of the ideal Π_{qp} , $\pi_{qp}(\bar{u}) = \pi_{qp}(\kappa_X \circ \bar{u})$, where $\kappa_X : X \rightarrow X^{**}$ is the canonical embedding. Finally, $\kappa_X \circ \bar{u} = \tilde{u}^{**}$, with $\tilde{u} : c_0 \rightarrow X$ defined via $\tilde{u}b = \sum_i b(a_i)x_i$.

To finish the proof, we need to show (in light of Theorem 3.1) that $\ell_1 \otimes_\alpha \ell_q$ satisfies the ℓ_q estimate. To this end, suppose we have a block-diagonal sequence $(u_i)_{i=1}^n$, and show that $\|\sum_i u_i\|_\alpha^q \sim \sum_i \|u_i\|_\alpha^q$. Abusing the notation slightly, we identify u_i with an operator from ℓ_∞^N to ℓ_q^N (where N is large enough), and identify $\|\cdot\|_\alpha$ with $\pi_{qp}(\cdot)$.

First show that $\|\sum_i u_i\|_\alpha^q \leq c^q \sum_i \|u_i\|_\alpha^q$, where c is a constant (depending on q). We have disjoint sets $(S_i)_{i=1}^n$ in $\{1, \dots, N\}$ so that $u_i e_j = 0$ for $j \notin S_i$. Therefore there exists a probability measure μ_i , supported on S_i , so that

$$\|u_i f\|^q \leq c_1^q \pi_{qp}(u_i)^q \|f\|_\infty^{q-p} \|f\|_{L_p(\mu_i)}^p$$

for any $f \in \ell_\infty^N$ (c_1 is a constant). Now define the probability measure μ on $\{1, \dots, N\}$:

$$\mu = \left(\sum_i \pi_{qp}(u_i)^q \right)^{-1} \sum_i \pi_{qp}(u_i)^q \mu_i.$$

For $f \in \ell_\infty^N$, set $f_i = f \chi_{S_i}$. Then the vectors $u_i f_i$ are disjointly supported in ℓ_q , and therefore,

$$\|(\sum_i u_i) f\|^q = \sum_i \|u_i f_i\|^q \leq c_1^q \sum_i \pi_{qp}(u_i)^q \|f_i\|_\infty^{q-p} \|f_i\|_{L_p(\mu_i)}^p \leq c_1^q \|f\|_\infty^{q-p} \sum_i \pi_{qp}(u_i)^q \|f_i\|_{L_p(\mu_i)}^p.$$

An easy calculation shows that

$$\|f_i\|_{L_p(\mu_i)}^p = \left(\sum_i \pi_{qp}(u_i)^q \right)^{-1} \sum_i \pi_{qp}(u_i)^q \|f_i\|_{L_p(\mu)}^p,$$

hence

$$\|(\sum_i u_i) f\|^q \leq c_1^q \left(\sum_i \pi_{qp}(u_i)^q \right) \|f\|_\infty^{q-p} \sum_i \|f_i\|_{L_p(\mu_i)}^p = c_1^q \left(\sum_i \pi_{qp}(u_i)^q \right) \|f\|_\infty^{q-p} \|f\|_{L_p(\mu)}^p.$$

Therefore, $\pi_{qp}(\sum_i u_i) \leq c \left(\sum_i \pi_{qp}(u_i)^q \right)^{1/q}$, for some universal constant c .

Next show that $\|\sum_i u_i\|_\alpha^q \geq c'^q \sum_i \|u_i\|_\alpha^q$, where c' is a constant. There exists a probability measure μ on $\{1, \dots, N\}$ so that, for any $f \in \ell_\infty^N$,

$$\|(\sum_u u_i f)\|^q \geq c_2^q \pi_{qp}(\sum_i u_i)^q \|f\|_\infty^{q-p} \|f\|_{L_p(\mu)}^p$$

For each i let $\alpha_i = \|\mu|_{S_i}\|_{\ell_1^N}$, and $\mu_i = \mu_i / \alpha_i$ (if $\alpha_i = 0$, then clearly $u_i = 0$). Then for any i , and any $f \in \ell_\infty^N$,

$$\|u_i f\|^q = \|(\sum_i u_i)(\chi_{S_i} f)\|^q \leq c_2^q \pi_{qp}(\sum_i u_i)^q \alpha_i^q \|f\|_\infty^{q-p} \|f\|_{L_p(\mu_i)}^p,$$

hence $\pi_{qp}(u_i) \leq c' \alpha_i^{1/q} \pi_{qp}(\sum_i u_i)$ (c' is a constant). As $\sum_i \alpha_i = 1$, we conclude that $\sum_i \pi_{qp}(u_i)^q \leq c'^q \pi_{qp}(\sum_i u_i)$. \blacksquare

4.3. Continuous fields. We refer the reader to [10, Chapter 10] for an introduction into continuous fields of Banach spaces. To set the stage, suppose K is a locally compact Hausdorff space (the *base space*), and $(X_t)_{t \in K}$ is a family of Banach spaces (the spaces X_t are called *fibers*). A *vector field* is an element of $\prod_{t \in K} X_t$. A linear subspace X of $\prod_{t \in K} X_t$ is called a *continuous field* if the following conditions hold:

- (1) For any $t \in K$, the set $\{x(t) : x \in X\}$ is dense in X_t .
- (2) For any $x \in X$, the map $t \mapsto \|x(t)\|$ is continuous, and vanishes at infinity.
- (3) Suppose x is a vector field so that, for any $\varepsilon > 0$ and any $t \in K$, there exist an open neighborhood $U \ni t$ and $y \in X$ for which $\|x(s) - y(s)\| < \varepsilon$ for any $s \in U$. Then $x \in X$.

Equipping X with the norm $\|x\| = \max_t \|x(t)\|$, we turn it into a Banach space.

In a fashion similar to Theorem 4.1, we prove:

Proposition 4.6. *Suppose K is a scattered metrizable space, X is a separable continuous vector field on K , so that, for every $t \in K$, the fiber X_t is subprojective. Then X is subprojective.*

Proof. Using one-point compactification if necessary (as in [10, 10.2.6]), we can assume that K is compact. As before, we assume that $K = [0, \lambda]$ (λ is a countable ordinal). We denote by $X_{(0)}$ the set of all $x \in X$ which vanish at λ . If $\nu \leq \lambda$, we denote by $X_{[\nu]}$ the set of all $x \in X_\lambda$ which vanish outside of $[0, \nu]$. By [10, Proposition 10.1.9], $x\chi_{[0, \nu]} \in X$ for any $x \in X$, hence $X_{[\nu]}$ is a Banach space. We then define the restriction operator $P_\nu : X \rightarrow X_{[\nu]}$. We denote by $Q_\nu : X \rightarrow X_\nu$ the operator of evaluation at ν .

We say that a countable ordinal λ has Property \mathcal{P} if, whenever X is a continuous separable vector field whose fibers are subprojective, then X is subprojective. Using transfinite induction, we prove that any countable ordinal has this property.

The base of induction is easy to handle. Indeed, when λ is finite, then X embeds into a direct sum of (finitely many) subprojective spaces X_ν . Now suppose, for the sake of contradiction, that λ is the smallest ideal failing Property \mathcal{P} . Note that λ is a limit ordinal. Indeed, otherwise it has an immediate predecessor λ_- , and X embeds into a direct sum of two subprojective spaces – namely, $X_{[\lambda_-]}$ and X_λ .

Suppose Y is a subspace of X , so that no subspace of Y is complemented in X . We shall achieve a contradiction once we show that Y contains a copy of c_0 .

By Proposition 2.2, Q_λ is strictly singular on Y . Passing to a smaller subsequence if necessary, we can assume that, Y has a basis $(y_i)_{i \in \mathbb{N}}$, so that (i) for any finite sequence (α_i) , $\|\sum_i \alpha_i y_i\| \geq \max_i |\alpha_i|/2$, and (ii) for any i , $\|Q_\lambda y_i\| < 10^{-4i}$. Consequently, for any $y \in \text{span}[y_j : j > i]$, $\|Q_\lambda y\| < 10^{-4i}$. Indeed, we can assume that y is a norm one vector with finite support, and write y as a finite sum $y = \sum_j \alpha_j y_j$. By the above, $|\alpha_i| \leq 2$ for every i . Consequently, $\|Q_\lambda y\| \leq \sum_j |\alpha_j| \|Q_\lambda y_j\| \leq 2 \sum_{j>i} 10^{-4j} < 10^{-4i}$.

Now construct a sequence $\nu_1 < \nu_2 < \dots < \lambda$ of ordinals, a sequence $1 = n_1 < n_2 < \dots$ or positive integers, and a sequence x_1, x_2, \dots of norm one vectors, so that (i) $x_j \in \text{span}[y_i : n_j \leq i < n_{j+1}]$, (ii) $\|P_{\nu_i} x_i\| < 10^{-4i}$, and (iii) $\|P_{\nu_{i+1}} x_i\| < 10^{-4i}$. To this end, recall that, by Proposition 2.2 again, $P_\nu|_Y$ is strictly singular for any $\nu < \lambda$. Pick an arbitrary $\nu_1 < \lambda$, and find a norm 1 vector $x_1 \in \text{span}[y_1, \dots, y_{n_2-1}]$ so that $\|P_{\nu_1} x_1\| < 10^{-4}$. We have $\|Q_\lambda x_1\| < 10^{-4}$. By continuity, we can find $\nu_2 > \nu_1$ so that $\|P_{\nu_2} x_1\| < 10^{-4}$. Next find a norm one $x_2 \in \text{span}[y_{n_2}, \dots, y_{n_3-1}]$ so that $\|P_{\nu_2} x_1\| < 10^{-8}$. Proceed further in the same manner.

We claim that the sequence (x_i) is equivalent to the canonical basis in c_0 . Indeed, for each i let $x_i'' = P_{\nu_i} x_i + P_{\nu_{i+1}} x_i$, and $x_i' = x_i - x_i''$. Since we are working with the sup norm, $\|x_i'\| = \|x_i\| = 1$ for any i . Furthermore, the elements x_i' are disjointly supported, hence, for any (α_i) finite sequence of scalars (α_i) , $\|\sum_i \alpha_i x_i'\| = \max_i |\alpha_i|$. By the triangle inequality,

$$\left| \left\| \sum_i \alpha_i x_i \right\| - \left\| \sum_i \alpha_i x_i' \right\| \right| \leq \sum_i |\alpha_i| \|x_i''\| < \max_i |\alpha_i| \sum_{i=1}^{\infty} 2 \cdot 20^{-4i} < 10^{-3} \max_i |\alpha_i|,$$

which yields the desired result. \blacksquare

To state a corollary of Proposition 4.6, recall that a C^* -algebra \mathcal{A} is *CCR* (or *liminal*) if, for any irreducible representation π of \mathcal{A} on a Hilbert space H , $\pi(\mathcal{A}) = \mathcal{K}(H)$. A C^* -algebra \mathcal{A} is *scattered* if every positive linear functional on \mathcal{A} is a sum of pure linear functionals ($f \in \mathcal{A}^*$ is called *pure* if it belongs to an extreme ray of

the positive cone of \mathcal{A}^*). For equivalent descriptions of scattered C^* -algebras, see e.g. [19, 20, 25].

Corollary 4.7. *Any separable scattered CCR C^* -algebra is subprojective.*

Proof. Suppose \mathcal{A} is a separable scattered CCR C^* -algebra. As shown in [33, Sections 6.1-3], the spectrum of a separable CCR algebra is a locally compact Hausdorff space. If, in addition, the algebra is scattered, then its spectrum $\hat{\mathcal{A}}$ is scattered as well [19, 20]. In fact, by the proof of [19, Theorem 3.1], $\hat{\mathcal{A}}$ is separable. It is easy to see that any separable locally compact Hausdorff space is metrizable.

By [10, Section 10.5], \mathcal{A} can be represented as a vector field over $\hat{\mathcal{A}}$, with fibers of the form $\pi(\mathcal{A})$, for irreducible representations π . As \mathcal{A} is CCR, the spaces $\pi(\mathcal{A}) = \mathcal{K}(H_\pi)$ (H_π being a separable Hilbert space) are subprojective. To finish the proof, apply Proposition 4.6. ■

The last corollary leads us to

Conjecture 4.8. A separable C^* -algebra is scattered if and only if it is subprojective.

It is known ([20], see also [25]) that a scattered C^* -algebra is GCR. However, it need not be CCR (consider the unitization of $\mathcal{K}(\ell_2)$).

5. SUBPROJECTIVITY OF SCHATTEN SPACES

In this section, we establish:

Proposition 5.1. *Suppose \mathcal{E} is a symmetric sequence space, not containing c_0 . Then $\mathfrak{C}_\mathcal{E}$ is subprojective if and only if \mathcal{E} is subprojective.*

The assumptions of this proposition are satisfied, for instance, if $\mathcal{E} = \ell_p$ ($1 \leq p < \infty$), or if \mathcal{E} is the Lorentz space $\mathfrak{l}(w, p)$ (see [26, Proposition 4.e.3]). However, not every symmetric sequence space is subprojective. Indeed, suppose \mathcal{E} is Pełczyński's universal space: it has an unconditional basis (u_i) so that any other unconditional basis is equivalent to its subsequence. As explained in [26, Section 3.b], \mathcal{E} has a symmetric basis. Fix $1 < p < q < 2$. Then the Haar basis in $L_p(0, 1)$ is unconditional, hence $L_p(0, 1)$ is isomorphic to a complemented subspace X of \mathcal{E} . It is well known that ℓ_q is contained in $L_p(0, 1)$. Call the corresponding subspace of \mathcal{E} by X' . Then no subspace of X' is complemented in \mathcal{E} : otherwise, $L_p(0, 1)$ would contain a complemented copy of ℓ_q , which is impossible.

For the proof, we need a technical result.

Proposition 5.2. *Suppose $\mathfrak{C}_\mathcal{E}$ is a symmetric sequence space, not containing c_0 . Suppose, furthermore, that $(z_n) \subset \mathfrak{C}_\mathcal{E}$ is a normalized sequence, so that, for every k , $\lim_n \|Q_k z_n\| = 0$. Then, for any $\varepsilon > 0$, $\mathfrak{C}_\mathcal{E}$ contains sequences (\tilde{z}_n) and (z'_n) , so that:*

- (1) (\tilde{z}_n) is a subsequence of (z_n) .
- (2) $\sum_n \|\tilde{z}_n - z'_n\| < \varepsilon$.
- (3) (z'_n) lies in the subspace Z of $\mathfrak{C}_\mathcal{E}$, with the property that (i) Z is 3-isomorphic to either ℓ_2 , \mathcal{E} , or $\ell_2 \oplus \mathcal{E}$, and (ii) Z is the range of a projection of norm not exceeding 3.

Proof. [3, Corollary 2.8] implies the existence of (\tilde{z}_n) and (z'_n) , so that (1) and (2) are satisfied, and $z'_k = a \otimes E_{1k} + b \otimes E_{k1} + c_k \otimes E_{kk}$ ($k \geq 2$). Thus, $z'_n \in Z = Z_r + Z_c + Z_d$, where $Z_r = \text{span}[a \otimes E_{1k} : k \geq 2]$ (the row component), $Z_c = \text{span}[b \otimes E_{k1} : k \geq 2]$ (the column component), and Z_d (the diagonal component) contains $c_k \otimes E_{kk}$, for any k . More precisely, we can write $c_k = u_k d_k v_k$, where u_k and v_k are unitaries, and d_k is diagonal. Then we set $Z_d = \text{span}[u_k E_{ii} v_k \otimes E_{kk} : i \in \mathbb{N}, k \geq 2]$.

It remains to build contractive projections P_r , P_c , and P_d onto Z_r , Z_c , and Z_d , respectively, so that $Z_c \cup Z_d \subset \ker P_r$, $Z_r \cup Z_d \subset \ker P_c$, and $Z_r \cup Z_c \subset \ker P_d$. Indeed, then $P = P_r + P_c + P_d$ is a projection onto $Z_r + Z_c + Z_d$, and the latter space is completely isomorphic to $Z_0 = Z_r \oplus Z_c \oplus Z_d$. The spaces Z_r , Z_c , and Z_d are either trivial (zero-dimensional), or isomorphic to ℓ_2 , ℓ_2 , and \mathcal{E} , respectively.

P_d is nothing but a coordinate projection, in the appropriate basis:

$$P_d(u_k E_{ij} v_\ell \otimes E_{k\ell}) = \begin{cases} u_k E_{ii} v_k \otimes E_{kk} & k = \ell \geq 2, i = j \\ 0 & \text{otherwise} \end{cases}$$

(for the sake of convenience, we set $u_1 = v_1 = I_{\ell_2}$). Next construct P_r (P_c is dealt with similarly). If $a = 0$, just take $P_r = 0$. Otherwise, let $a' = a/\|a\|$, and find $f \in \mathfrak{C}_{\mathcal{E}}^*$ so that $\|f\| = 1 = \langle f, a' \rangle$. For $x = \sum_{k,\ell} x_{k\ell} \otimes E_{k\ell}$, define

$$P_r x = a' \otimes \sum_{\ell \geq 2} \langle f, x_{1\ell} \rangle E_{1\ell},$$

hence $\|P_r x\|_{\mathcal{E}}^2 = \sum_{\ell \geq 2} |\langle f, x_{1\ell} \rangle|^2$. It remains to show $\|P_r x\| \leq \|x\|$. This inequality is obvious when $P_r x = 0$. Otherwise, set, for $\ell \geq 2$,

$$\alpha_\ell = \frac{\overline{\langle f, x_{1\ell} \rangle}}{(\sum_{\ell \geq 2} |\langle f, x_{1\ell} \rangle|^2)^{1/2}},$$

$y = I_{\ell_2} \otimes \sum_{\ell \geq 2} \alpha_\ell E_{\ell 1}$, and $z = I_{\ell_2} \otimes E_{11}$. Then $\|y\|_\infty = (\sum_{\ell \geq 2} |\alpha_\ell|^2)^{1/2} = 1 = \|z\|_\infty$, and $zxy = \sum_{\ell \geq 2} \alpha_\ell x_{1\ell} \otimes E_{11}$. Therefore,

$$\begin{aligned} \|P_r x\|_{\mathcal{E}} &= \langle f, \sum_{\ell \geq 2} \alpha_\ell x_{1\ell} \rangle \leq \left\| \sum_{\ell \geq 2} \alpha_\ell x_{1\ell} \right\|_{\mathcal{E}} \\ &= \left\| \sum_{\ell \geq 2} \alpha_\ell x_{1\ell} \otimes E_{11} \right\|_{\mathcal{E}} = \|zxy\|_{\mathcal{E}} \leq \|z\|_\infty \|x\|_{\mathcal{E}} \|y\|_\infty = \|x\|_{\mathcal{E}}, \end{aligned}$$

which is what we need. ■

Proof of Proposition 5.1. The space $\mathfrak{C}_{\mathcal{E}}$ contains an isometric copy of \mathcal{E} , hence the subprojectivity of $\mathfrak{C}_{\mathcal{E}}$ implies that of \mathcal{E} . To prove the converse, suppose \mathcal{E} is subprojective, and Z_0 is a subspace of $\mathfrak{C}_{\mathcal{E}}$, and show that it contains a further subspace Z , complemented in $\mathfrak{C}_{\mathcal{E}}$. To this end, find a normalized sequence $(z_n) \subset Z_0$, so that $\lim_n \|Q_k z_n\| = 0$ for every k . By Proposition 5.2, (z_n) has a subsequence (z'_n) , contained in a subspace Z_1 , which is complemented in $\mathfrak{C}_{\mathcal{E}}$, and isomorphic either to \mathcal{E} , ℓ_2 , or $\mathcal{E} \oplus \ell_2$. By Proposition 2.1, Z_1 is subprojective, hence $\text{span}[z'_n : n \in \mathbb{N}]$ contains a subspace complemented in Z_1 , hence also in $\mathfrak{C}_{\mathcal{E}}$. ■

As a consequence we obtain:

Proposition 5.3. *The predual of a von Neumann algebra \mathcal{A} is subprojective if and only if \mathcal{A} is purely atomic.*

We say that \mathcal{A} is *purely atomic* if any projection in it has an atomic subprojection. It is easy to see that this happens if and only if $\mathcal{A} = (\sum_i B(H_i))_\infty$. The “if” direction is easy. Conversely, if \mathcal{A} is purely atomic, denote by $(e_i)_{i \in I}$ a maximal collection of mutually non-equivalent atomic projections in \mathcal{A} . Denote by $z(p)$ the central cover of p . Then $z(e_i)z(e_j) = 0$ if $i \neq j$, and $\sum_i z(e_i) = 1$. Consequently, $\mathcal{A} = \sum_i z(e_i)\mathcal{A}$. For a fixed i , let $(f_j)_{j \in J(i)}$ be a maximal family of mutually orthogonal atomic projections, so that e_i is one of these projections. The f_j ’s have the same central cover (namely, $z(e_i)$), hence they are all equivalent to e_i . Furthermore, $z(e_i) = \sum_{j \in J(i)} f_j$, hence $z(e_i)\mathcal{A}$ is isomorphic to $B(\ell_2(J(i)))$.

Proof. If a von Neumann algebra \mathcal{A} is not purely atomic, then, as explained in [30, Section 1], \mathcal{A}_* contains a (complemented) copy of $L_1(0,1)$. This establishes the “only if” implication of Proposition 5.3. Conversely, if \mathcal{A} is purely atomic, then \mathcal{A}_* is isometric to a (contractively complemented) subspace of $\mathfrak{C}_1(H)$, and the latter is subprojective. ■

6. p -CONVEX AND p -DISJOINTLY HOMOGENEOUS BANACH LATTICES

We say that X is *p -disjointly homogeneous* (*p -DH* for short) if every disjoint normalized sequence contains a subsequence equivalent to the standard basis of ℓ_p .

For the sake of completeness we present a proof of the following statement (see [15, 4.11, 4.12]).

Proposition 6.1. *Let X be a p -convex. Then every subspace, spanned by a disjoint sequence equivalent to the canonical basis of ℓ_p , is complemented.*

Proof. Let $(x_k) \subset X$ be a disjoint normalized sequence. Since X is DH, by passing to a subsequence, (x_k) is an ℓ_p basic sequence. Then, in the p -concavification $X_{(p)}$ the disjoint sequence (x_k^p) is an ℓ_1 basic sequence. Therefore, there exists a functional $x^* \in [(x_k^p)]$ such that $x^*(x_k^p) = 1$ for all k . By the Hahn-Banach Theorem x^* can be extended to a positive functional in $X_{(p)}^*$. Define a seminorm $\|x\|_p = (x^*(|x^p|))^{\frac{1}{p}}$ on X . Denote by \mathcal{N} the subset of X on which this seminorm is equal to zero. Clearly, \mathcal{N} is an ideal, therefore, the quotient space $\tilde{X} = X/\mathcal{N}$ is a Banach lattice, and the quotient map $Q : X \rightarrow \tilde{X}$ is an orthomorphism. With the defined seminorm \tilde{X} is an abstract L_p -space, and the disjoint sequence $Q(x_k)$ is normalized. Therefore it is an ℓ_p basic sequence that spans a complemented subspace (in particular, Q is an isomorphism when restricted to $[x_k]$). Let \tilde{P} be a projection from \tilde{X} onto $[Q(x_k)]$. Then $P = Q^{-1}\tilde{P}Q$ is a projection from X onto $[x_k]$. ■

Proposition 6.2. *Let X be a p -convex, p -disjointly homogeneous Banach lattice ($p \geq 2$). Then any subspace of X contains a complemented copy of either ℓ_p or ℓ_2 . Consequently, X is subprojective.*

Proof. First, note that X is order continuous. Let $M \subseteq X$ be an infinite dimensional separable subspace. Then there exists a complemented order ideal in X with a weak unit that contains M . Therefore, without loss of generality, we may assume that X

has a weak unit. Then there exists a probability measure μ [21, p. 14] such that we have continuous embeddings

$$L_\infty(\mu) \subseteq X \subseteq L_p(\mu) \subseteq L_2(\mu) \subseteq L_1(\mu).$$

Consequently, there exists a constant $c_1 > 0$ so that $c_1 \|x\|_p \leq \|x\|$ for any $x \in X$.

By the proof of [27, Proposition 1.c.8], one of the following holds:

Case 1. M contains an almost disjoint bounded sequence. By Proposition 6.1 M contains a copy of ℓ_p complemented in X .

Case 2. The norms $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent on M . Thus, there exists $c_2 > 0$ so that, for any $y \in M$,

$$c_2 \|y\|_2 \geq c_2 \|y\|_1 \geq \|y\| \geq c_1 \|y\|_p \geq c_1 \|y\|_2.$$

In particular, M is embedded into $L_2(\mu)$ as a closed subspace. The orthogonal projection from $L_2(\mu)$ onto M then defines a bounded projection from X onto M . ■

The preceding result implies that Lorentz space $\Lambda_{p,W}(0,1)$ is subprojective since it is p -DH and p -convex ($p \geq 1$), see [14, Theorem 3] and [22]. Note that, originally, the subprojectivity of $\Lambda(p, W)$ ($p \geq 2$) was observed in [14, Remark 5.7].

7. LATTICE-VALUED ℓ_p SPACES

If X is a Banach lattice, and $1 \leq p < \infty$, denote by $\widetilde{X(\ell_p)}$ the completion of the space of all finite sequences (x_1, \dots, x_n) (with $x_i \in X$), equipped with the norm $\|(x_1, \dots, x_n)\| = \|(\sum_i |x_i|^p)^{1/p}\|$, where

$$(\sum_i |x_i|^p)^{1/p} = \sup \left\{ \left| \sum_i \alpha_i x_i \right| : \sum_i |\alpha_i|^{p'} \leq 1 \right\}, \text{ with } \frac{1}{p} + \frac{1}{p'} = 1.$$

See [27, pp. 46-48] for more information. We have:

Proposition 7.1. *Suppose X is a subprojective separable space, with the lattice structure given by an unconditional basis, and $1 \leq p < \infty$. Then $\widetilde{X(\ell_p)}$ is subprojective.*

Proof. To show that any subspace $Y \subset \widetilde{X(\ell_p)}$ has a further subspace Z , complemented in $\widetilde{X(\ell_p)}$, let x_1, x_2, \dots and e_1, e_2, \dots be the canonical bases in X and ℓ_p , respectively. Then the elements $u_{ij} = x_i \otimes e_j$ form an unconditional basis in $\widetilde{X(\ell_p)}$, with

$$(7.1) \quad \left\| \sum a_{ij} u_{ij} \right\| = \left\| \sum_i \left(\sum_j |a_{ij}|^p \right)^{1/p} x_i \right\|_X = \left\| \sum_i \left(\sup_{\sum_j |\alpha_j|^{p'} \leq 1} \left| \sum_j \alpha_j a_{ij} \right| \right) x_i \right\|_X.$$

Let P_n be the canonical projection onto $\text{span}[u_{ij} : 0 \leq i \leq n, j \in \mathbb{N}]$, and set $P_n^\perp = I - P_n$. The range of P_n is isomorphic to ℓ_p , hence, if $P_n|_Y$ is not strictly singular for some n , we are done, by Corollary 2.3. If $P_n|_Y$ is strictly singular for every n , find a normalized sequence (y_i) in Y , and $1 = n_1 < n_2 < \dots$, so that $\|P_{n_i} y_i\|, \|P_{n_{i+1}}^\perp y_i\| < 100^{-i}/2$. By small perturbation, it remains to prove the following: if $y_i = P_{n_i}^\perp P_{n_{i+1}} y_i$,

then $\text{span}[y_i : i \in \mathbb{N}]$ contains a subspace, complemented in $\widetilde{X(\ell_p)}$. Further, we may assume that for each i there exists M_i so that we can write

$$y_i = \sum_{n_i < k \leq n_{i+1}, 1 \leq j \leq M_i} a_{kj} u_{kj}.$$

For each $k \in [n_i + 1, n_{i+1}]$ (and arbitrary $i \in \mathbb{N}$) find a finite sequence $(\alpha_{kj})_{j=1}^{M_i}$ so that $\sum_j |\alpha_{kj}|^{p'} = 1$, and $|\sum_j \alpha_{kj} a_{kj}| = (\sum_j |a_{kj}|^p)^{1/p}$. Define $U : \widetilde{X(\ell_p)} \rightarrow X : u_{kj} \mapsto \alpha_{kj} a_{kj} x_k$. By (7.1), U is a contraction, and $U|_{\text{span}[y_i : i \in \mathbb{N}]}$ is an isometry. To finish the proof, recall that X is subprojective, and apply Corollary 2.3. \blacksquare

Remark 7.2. Using similar methods, one can prove: if K is a compact metrizable space, and $1 \leq p < \infty$, then $C(\widetilde{K})(\ell_p)$ is subprojective.

Recall that, for a Banach space X , we denote by $\text{Rad}(X)$ the completion of the finite sums $\sum_n r_n x_n$ (r_1, r_2, \dots are Rademacher functions, and $x_1, x_2, \dots \in X$) in the norm of $L_1(X)$ (equivalently, by Khintchine-Kahane Inequality, in the norm of $L_p(X)$). If X has an unconditional basis (x_i) and finite cotype, then $\text{Rad}(X)$ is isomorphic to $\widetilde{X(\ell_2)}$ (here we can view X as a Banach lattice, with the order induced by the basis (x_i)). Indeed, by [27, Section 1.f], X is q -concave, for some q . An array (a_{mn}) can be identified both with an element of $\text{Rad}(X)$ (with the norm $\int_0^1 \|\sum_m \sum_n a_{mn} r_n x_m\|$), and with an element of $\widetilde{X(\ell_2)}$ (with the norm $\|\sum_m (\sum_n |a_{mn}|^2)^{1/2} x_m\|$). Then

$$\begin{aligned} D \|\sum_m (\sum_n |a_{mn}|^2)^{1/2} x_m\| &\leq \|\sum_m \int_0^1 |\sum_n a_{mn} r_n| x_m\| = \|\int_0^1 |\sum_m \sum_n a_{mn} r_n x_m|\| \\ &\leq \int_0^1 \|\sum_m \sum_n a_{mn} r_n x_m\| \leq (\int_0^1 \|\sum_m \sum_n a_{mn} r_n x_m\|^q)^{1/q} \\ &\leq M_q \|(\int_0^1 |\sum_m \sum_n a_{mn} r_n x_m|^q)^{1/q}\| \leq M_q \|\sum_m (\int_0^1 |\sum_n a_{mn} r_n|^q)^{1/q} x_m\| \\ &\leq C M_q \|\sum_m (\sum_n |a_{mn}|^2)^{1/2} x_m\|, \end{aligned}$$

where M_q is a q -concavity constant, while D and C come from Khintchine's inequality. Thus, we have proved:

Proposition 7.3. *If X is a subprojective space with an unconditional basis and non-trivial cotype, then $\text{Rad}(X)$ is subprojective.*

Remark 7.4. By [24, Theorem 2.3], if X is a non-atomic order continuous Banach lattice with an unconditional basis, then $\widetilde{X(\ell_2)}$ is isomorphic to X . Furthermore, if X is a non-atomic Banach lattice with an unconditional basis and non-trivial cotype, then $\text{Rad}(X)$ is isomorphic to X . Indeed, non-trivial cotype implies non-trivial lower estimate [27, p. 100], which, by [28, Theorem 2.4.2], implies order continuity. Therefore, X is isomorphic to $\widetilde{X(\ell_2)}$, which, in turn, is isomorphic to $\text{Rad}(X)$.

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DEPT. OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA IL 61801, USA

E-mail address: oikhberg@illinois.edu

DEPT. OF MATHEMATICAL AND STATISTICAL SCIENCES, UNIVERSITY OF ALBERTA EDMONTON, ALBERTA T6G 2G1, CANADA

E-mail address: espinu@ualberta.ca